Math 247A Lecture 22 Notes

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1 The Fractional Chain Rule

1.1 Proof of the fractional chain rule

Theorem 1.1 (Fractional chain rule, Christ-Weinstein, 1991)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be such that*

$$|F(u) - F(v)| \leq |u - v|[G(u) + G(v)],$$

where $G : \mathbb{C} \to [0, \infty)$. Then for $0 < s < 1$, $1 < p, p_1 < \infty$, $1 < p_2 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$\||\nabla|^s (F \circ u)\|_p \lesssim \||\nabla|^s u\|_{p_1} \cdot \|G \circ u\|_{p_2}.$$

Example 1.1. Consider some nonlinear interaction: Let $F(u) = |u|^p u$, where $p > 0$. Then

$$|F(u) - F(v)| \lesssim |u - v|[|u|^p + |v|^p],$$

so we get a bound.

Proof. Last time, we showed that

$$\||\nabla|^s (F \circ u)\|_p \sim \left\| \sqrt{\sum N^{2s} |P_N(F \circ u)|^2} \right\|_p .$$

Let's calculate

$$[P_N(f \circ u)](x) = \int N^d \psi^\vee (Nu)(F \circ u)(x - y)\, dy$$

We want to isolate u in this expression. We will use the locally Lipschitz condition. Since $\int \psi^\vee \, dy = \psi(0) = 0$,

$$= \int N^d \psi_\vee (Ny)[(F \circ u)(x - y) - (F \circ u)(x)]\, dy.$$

So we have

$$|P_N(F\circ u)|(x) \le \int N^d|\psi^\vee(Ny)|\cdot|u(x-y)-u(x)|[(G\circ u)(x-y)+(G\circ u)(x)]\,dy$$

We expect cancellation in the u terms at low frequencies. So we decompose

$$|u(x-y)-u(x)| \le \underbrace{|u_{>N}(x-y)|}_{I} + \underbrace{|u_{>N}(x)|}_{II} + \underbrace{\sum_{k\le N}|u_k(x-y)-u_k(x)|}_{III}.$$

Let's consider the contribution of I:

$$\int N^d|\psi^\vee(Ny)|u_{>N}(x-y)|[(G\circ u)(x-y)+(G\circ u)(x)]\,dy$$

We can bound this using the maximal function. We have $\int N^d|\psi^\vee(Ny)||g(x-y)|\,dy \lesssim \int_{|y|\le 1/N}|g(x-y)|\,dy + \sum_{R\in 2^{\mathbb{N}}}\int_{R/N\le|y|\le 2R/N} N^d\frac{1}{R^{2d}}|g(x-y)|\,dy \lesssim \frac{1}{|B(0,1/N)|}\int_{B(0,1/N)}|g(x-y)|\,dy + \cdots$.

$$\begin{aligned}&\lesssim M(u_{\ge N}(G\circ u))(x) + M(u_{>N})(x)(G\circ u)(x)\\ &\lesssim M(u_{>N}(G\circ u))(x) + M(u_{>N})(x)M(G\circ u)(x).\end{aligned}$$

This contributes the following to the original estimate:

$$\begin{aligned}&\left\|\sqrt{\sum N^{2s}|M(u_{>N}(G\circ u))|^2}\right\|_p + \left\|\sqrt{\sum N^{2s}|M(u_{>N})M(G\circ u)|^2}\right\|_p\\ &\quad\lesssim \left\|\sqrt{\sum |M(N^s u_{>N})(G\circ u)|^2}\right\|_p + \left\|M(G\circ u)\sqrt{\sum|M(N^s u_{>N})|^2}\right\|_p\end{aligned}$$

Using our bounds for the vector-valued maximal function and Hölder,

$$\begin{aligned}&\lesssim \left\|\sqrt{\sum_N |N^s u_{>N}|^2}(G\circ u)\right\| + \left\|\sqrt{\sum|N^s u_{>N}|^2}\right\|_{p_1}\|M(G\circ u)\|_{p_2}\\ &\lesssim \||\nabla|^s u\|_{p_1}\cdot\|G\circ u\|_{p_2}.\end{aligned}$$

This is an acceptable contribution for what we want to prove.

Let's look at what II. To $P_N(F\circ u)$, this contributes

$$\begin{aligned}&\int N^d|\psi^\vee(Ny)|u_{>N}(x)|[(G\circ u)(x-y)+(G\circ u)(x)]\,dy\\ &\quad\lesssim |u_{>N}(x)|M(G\circ u)(x) + |u_{>N}(x)|(G\circ u)(x)\\ &\quad\lesssim M(u_{>n})(x)M(G\circ u)(x).\end{aligned}$$

As before, the contribution of II to the right hand side of the original estimate is acceptable.

We turn to III. We claim that

$$|u_k(x-y) - u_k(x)| \lesssim k|y| \cdot [M(u_k)(x-y) + Mu_k(x)]$$

We split into cases:

1. $k|y| > 1$: Then

$$\begin{aligned} |u_k(x-y) - u_k(x)| &\le |\widetilde{P}_k u_k|(x-y) + |\widetilde{P}_k u_k|(x) \\ &\lesssim (Mu_k)(x-y) + (Mu_k)(x). \end{aligned}$$

2. $k|y| \le 1$:

$$\begin{aligned} |u_k(x-y) - u_k(x)| &= \left| \int k^d \widetilde{\psi}^\vee(kz)[u_k(x-y-z) - u_k(x-z)]\, dz \right| \\ &= \left| \int k^d [\widetilde{\psi}^\vee(k(z-y)) - \widetilde{\psi}^\vee(kz)] u_k(x-z)\, dz \right| \end{aligned}$$

Using the fundamental theorem of calculus,

$$\begin{aligned} &= \int k^d k|y| \int_0^1 \underbrace{|\nabla \widetilde{\psi}^\vee|(kz - \theta k y)}_{\lesssim 1/\langle kz-\theta ky\rangle^{2d} \lesssim 1/\langle kz\rangle^{2d}} |u_k(x-z)|\, d\theta\, dz \\ &\lesssim k|y| \cdot (Mu_k)(x), \end{aligned}$$

proving the claim.

To $P_N(f \circ u)$, the term III contributes

$$\begin{aligned} &\int N^d |\psi^\vee(Ny)| \sum_{K \le N} K|y| [(Mu_k)(x-y) - M(u_k)(x)] \cdot [(G\circ u)(x-y) + (G \circ u)(x)]\, dy \\ &\lesssim \sum_{k \le N} \frac{k}{N} \int N^d \frac{N|y|}{\langle N|y|\rangle^{3d}} [(Mu_k)(x-y) + (Mu_k)(x)] \cdot [(G\circ u)(x-y) + (G\circ u)(x)]\, dy \\ &\lesssim \sum_{k \le N} \frac{k}{N} \cdot [M((Mu_k)\cdot (G\circ u))(x) + M(Mu_k)(x) \cdot M(G \circ u)(x)]. \end{aligned}$$

The contribution of III to the right hand side of the original estimate is

$$\lesssim \left\| \sqrt{\sum_N N^{2s} \left| \sum_{k \le N} \frac{k}{N} M((Mu_k)(G\circ u)) \right|^2} \right\|_p + \left\| \sqrt{N^{2s} \left| \sum_{k\le N} \frac{k}{N} M(Mu_k) \cdot M(G \circ u) \right|^2} \right\|_p$$

Both cases have terms like

$$\begin{aligned}
\sum_N N^{2s} \left| \sum_{k \le N} \frac{k}{N} c_k \right|^2 &\le 2 \sum_{k \le L \le N} N^{2s} \frac{kL}{NN} |c_k||c_L| \\
&\lesssim \sum_{k \le L} L^{2s} \frac{k}{L} |c_k||c_L| \\
&\lesssim \sum_{k \le L} \left(\frac{k}{L} \right)^{1-s} k^s |c_k| L^s |c_L|
\end{aligned}$$

By Cauchy-Schwarz (or Schur's test),

$$\begin{aligned}
&\lesssim \sqrt{\sum_k k^{2s} |c_k|^2} \sqrt{\sum_L L^{2s} |c_L|^2} \\
&\lesssim \sum_N N^{2s} |c_N|^2.
\end{aligned}$$

And we use our maximal function bounds to finish the proof. □